

Note on Series for $\ln 2$

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When I was a high-school senior visiting CWRU, Prof. A. J. Lohwater teased me with the following proof. Start with:

$$\ln 2 = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots$$

I've colored the terms to make subsequent rearrangements easier to see. Multiply both sides by two:

$$2 \ln 2 = \frac{2}{1} - \frac{2}{2} + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} + \frac{2}{7} - \frac{2}{8} + \dots$$

Rearrange the terms by odd and even denominators:

$$2 \ln 2 = \frac{2}{1} + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \dots - \frac{2}{2} - \frac{2}{4} - \frac{2}{6} - \frac{2}{8} - \dots$$

Simplify and extend the second subseries:

$$2 \ln 2 = \frac{2}{1} + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \dots - \frac{1}{1} - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} - \dots$$

Rearrange the terms:

$$2 \ln 2 = \left(\frac{2}{1} - \frac{1}{1}\right) - \frac{1}{2} + \left(\frac{2}{3} - \frac{1}{3}\right) - \frac{1}{4} + \left(\frac{2}{5} - \frac{1}{5}\right) - \frac{1}{6} + \left(\frac{2}{7} - \frac{1}{7}\right) - \frac{1}{8} + \dots$$

Simplify the parenthesized expressions:

$$2 \ln 2 = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots$$

The right side is the series for $\ln 2$, thus:

$$2 \ln 2 = \ln 2$$

Divide both sides by $\ln 2$:

$$2 = 1$$

Q.E.D.